Final Project

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Contents

1 First Component

Given: Consider a crane that moves along an one-dimensional track. It behaves as a friction less cart with mass M actuated by an external force F that constitutes the input of the system. There are two loads suspended from cables attached to the crane. The loads have mass $m1$ and $m2$, and the lengths of the cables are $l1$ and $l2$, respectively. The following figure depicts the crane and associated variables used throughout this project.

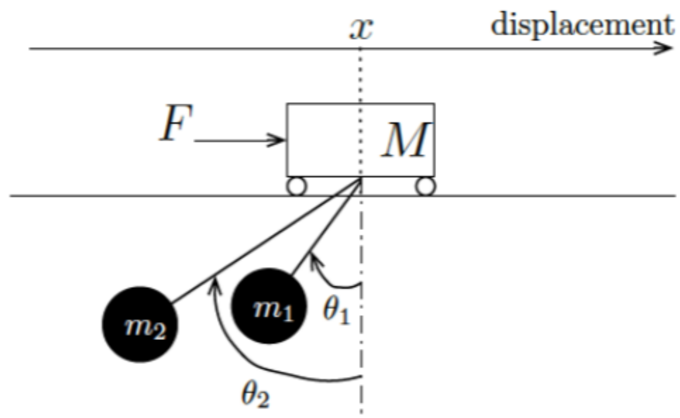

Figure 1: Given System

1.1 Equations of Motion

To find the equations of motion of the system, we have used the Euler-Lagrange's method. The first step of which was to get the position the three masses present in the system, with respect to distance and the angles of the pendulum.

For the first mass, i.e M, the position is

$$\text{Position of mass } M = (x, 0)$$

And to calculate the position of mass m_1 and m_2, we find the x and y components of the position as shown below.

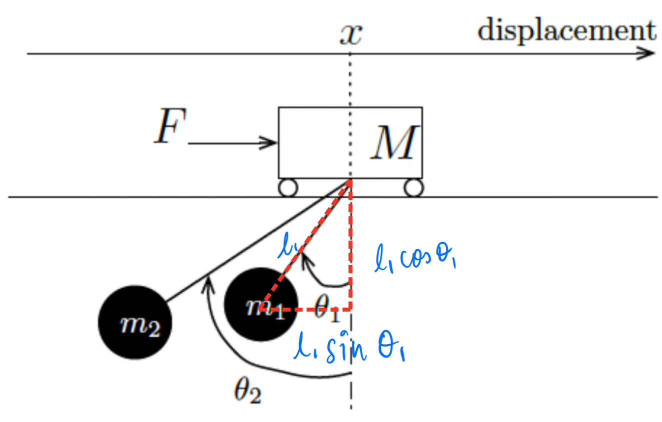

Figure 2: Positions components for the mass m_1

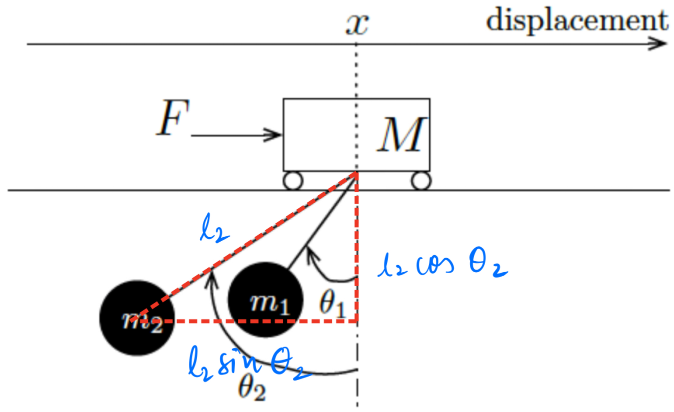

Figure 3: Position components for the mass m_2

Hence, their positions can be given as,

$$\text{Position of mass } m_1 = (x - l_1 \sin\theta_1, -l_1 \cos\theta_1)$$

$$\text{Position of mass } m_2 = (x - l_2 \sin\theta_2, -l_2 \cos\theta_2)$$

From the above equations, the velocities of the masses can be calculated by derivating the position components with respect to t, as shown below,

$$\text{Velocity of mass } M = V_M = (\dot{x}, 0)$$
$$\text{Velocity of mass } m_1 = V_{m_1} = \left(\dot{x} - l_1\dot{\theta}_1 \cos\theta_1,\ l_1\dot{\theta}_1 \sin\theta_1\right)$$
$$\text{Velocity of mass } m_2 = V_{m_2} = \left(\dot{x} - l_2\dot{\theta}_2 \cos\theta_2,\ l_2\dot{\theta}_2 \sin\theta_2\right)$$

To find the kinetic energy of each mass, we use the formula $\frac{1}{2}mv^2$ Which gives us,

$$\text{Kinetic Energy of mass } M = K.E_1 = \frac{1}{2}MV_M^2 = \frac{1}{2}M\dot{x}^2$$
$$\text{Kinetic Energy of mass } m_1 = K.E_2 = \frac{1}{2}m_1V_{m_1}^2 = \frac{1}{2}m_1\left(\dot{x}^2 + l_1^2\dot{\theta}_1^2 - 2l_1\dot{x}\dot{\theta}_1\cos\theta_1\right)$$
$$\text{Kinetic Energy of mass } m_2 = K.E_3 = \frac{1}{2}m_2V_{m_2}^2 = \frac{1}{2}m_2\left(\dot{x}^2 + l_2^2\dot{\theta}_2^2 - 2l_2\dot{x}\dot{\theta}_2\cos\theta_2\right)$$

To find the kinetic energies of the masses, we used the formula mgh, and we get,

$$\text{Potential Energy of mass } M = P.E_1 = 0$$

$$\text{Potential Energy of mass } m_1 = P.E_2 = -m_1gl_1\cos\theta_1$$

$$\text{Potential Energy of mass } m_2 = P.E_3 = -m_2gl_2\cos\theta_2$$

The Lagrangian is calculated as:

$$\mathcal{L} = K - P$$
$$\mathcal{L} = K.E_1 + K.E_2 + K.E_3 - (P.E_1 + P.E_2 + P.E_3)$$

Substituting the values of energies calculated, we get,

$$\mathcal{L}\left(x,\theta_1,\theta_2\right) = \frac{1}{2}\left[\dot{x}^2\left(M+m_1+m_2\right)+m_1l_1^2\dot{\theta}_1^2+m_1l_2^2\dot{\theta}_2^2\right]-m_1l_1\dot{x}\dot{\theta}_1\cos\theta_1-m_2l_2\dot{x}\dot{\theta}_2\cos\theta_2$$
$$+m_1gl_1\cos\theta_1+m_2gl_2\cos\theta_2$$

To get the equations of motion, we use the following,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}-\frac{\partial L}{\partial \dot{q}_i}$$

Where q is a generalized parameter, which are x, θ_1 and θ_2 for us. Hence to get the first equation of motion, we use the x, which gives us,

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \dot{x}\left(M+m_1+m_2\right)-m_1l_1\dot{\theta}_1\cos\theta_1-m_2l_2\dot{\theta}_2\cos\theta_2$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \ddot{x}\left(M+m_1+m_2\right)-m_1l_1\left(\ddot{\theta}_1\cos\theta_1-\dot{\theta}_1^2\sin\theta_1\right)-m_2l_2\left(\ddot{\theta}_2\cos\theta_2-\dot{\theta}_2^2\sin\theta_2\right)$$
$$\frac{\partial \mathcal{L}}{\partial x}=0$$
$$F_1=\left(M+m_1+m_2\right)\ddot{x}-m_1l_1\left(\ddot{\theta}_1\cos\theta_1-\dot{\theta}_1^2\sin\theta_1\right)-m_2l_2\left(\ddot{\theta}_2\cos\theta_2-\dot{\theta}_2^2\sin\theta_2\right)$$

Next, the second equation of motion is calculated using θ_1 as,

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}=m_1l_1^2\dot{\theta}_1-m_1l_1\dot{x}\cos\theta_1$$
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}=m_1l_1^2\ddot{\theta}_1-m_1l_1\left(\ddot{x}\cos\theta_1-\dot{x}\dot{\theta}_1\sin\theta_1\right)$$
$$\frac{\partial \mathcal{L}}{\partial \theta_1}=m_1l_1\dot{x}\dot{\theta}_1\sin\theta_1-m_1gl_1\sin\theta_1$$
$$F_2=m_1l_1^2\ddot{\theta}_1-m_1l_1\ddot{x}\cos\theta_1+m_1gl_1\sin\theta_1$$

and the third equation of motion is,

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}=m_2l_2^2\dot{\theta}_2-m_2l_2\dot{x}\cos\theta_2$$
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}=m_2l_2^2\ddot{\theta}_2-m_2l_2\left(\ddot{x}\cos\theta_2-\dot{x}\dot{\theta}_2\sin\theta_2\right)$$
$$\frac{\partial \mathcal{L}}{\partial \theta_2}=m_2l_2\dot{x}\dot{\theta}_2\sin\theta_2-m_2gl_2\sin\theta_2$$
$$F_3=m_2l_2^2\ddot{\theta}_2-m_2l_2\ddot{x}\cos\theta_2+m_2gl_2\sin\theta_2$$

Now, as the only force acting on the system is in the x direction, hence the other forces would be 0, which means,

$$F_1=F$$
$$F_2=F_3=0$$

Which gives us,

$$F=\left(M+m_1+m_2\right)\ddot{x}-m_1l_1\left(\ddot{\theta}_1\cos\theta_1-\dot{\theta}_1^2\sin\theta_1\right)-m_2l_2\left(\ddot{\theta}_2\cos\theta_2-\dot{\theta}_2^2\sin\theta_2\right) \tag{1}$$

$$0 = m_1 l_1^2 \ddot{\theta}_1 - m_1 l_1 \ddot{x} \cos\theta_1 + m_1 g l_1 \sin\theta_1 \quad (2)$$

$$0 = m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2 \ddot{x} \cos\theta_2 + m_2 g l_2 \sin\theta_2 \quad (3)$$

From equations 2 and 3, we get the equations of $\ddot{\theta_1}$ and $\ddot{\theta_2}$ as such,

$$\ddot{\theta}_1 = \frac{1}{l_1} \left(\ddot{x} \cos\theta_1 - g \sin\theta_1\right) \quad (4)$$

$$\ddot{\theta}_2 = \frac{1}{l_2} \left(\ddot{x} \cos\theta_2 - g \sin\theta_2\right) \quad (5)$$

Substituting the values of $\ddot{\theta_1}$ and $\ddot{\theta_2}$ in equation 1, we get

$$\ddot{x} = \frac{F - m_1 g \sin\theta_1 \cos\theta_1 - m_2 g \sin\theta_2 \cos\theta_2 - m_1 l_1 \sin\theta_1 \dot{\theta_1}^2 - m_2 l_2 \sin\theta_2 \dot{\theta_2}^2}{M + m_1 + m_2 - m_1 \cos^2\theta_1 - m_2 \cos^2\theta_2}$$

Using the above calculated value of $\ddot{x}$, and substituting in equations 4 and 5, we get

$$\ddot{\theta}_1 = \frac{1}{l_1} \left(\left(\frac{F - m_1 g \sin\theta_1 \cos\theta_1 - m_2 g \sin\theta_2 \cos\theta_2 - m_1 l_1 \sin\theta_1 \dot{\theta_1}^2 - m_2 l_2 \sin\theta_2 \dot{\theta_2}^2}{M + m_1 + m_2 - m_1 \cos^2\theta_1 - m_2 \cos^2\theta_2} \right) \cos\theta_1 - g \sin\theta_1 \right) \quad (6)$$

$$\ddot{\theta}_2 = \frac{1}{l_2} \left(\left(\frac{F - m_1 g \sin\theta_1 \cos\theta_1 - m_2 g \sin\theta_2 \cos\theta_2 - m_1 l_1 \sin\theta_1 \dot{\theta_1}^2 - m_2 l_2 \sin\theta_2 \dot{\theta_2}^2}{M + m_1 + m_2 - m_1 \cos^2\theta_1 - m_2 \cos^2\theta_2} \right) \cos\theta_2 - g \sin\theta_2 \right) \quad (7)$$

1.2 Linearization of the system in state space form

Now we begin to linearlize our system in order to represent in standard state space representation. Before representing in the state space form we need to make some assumptions i.e the angles θ_1 and θ_2 are considered to be small and the following higher order derivatives will be closer to zero. We are linearizing our system by following this approximation.This approximation is called as small angle approximation. These can be seen below :

$$\begin{aligned}
\sin\theta_1 &\simeq \theta_1 \\
\sin\theta_2 &\simeq \theta_2 \\
\sin\theta^2 &\simeq 0 \\
\cos\theta_1 &\simeq 1 \\
\cos\theta_1 &\simeq 0 \\
\cos\theta^2 &\simeq 1 \\
\frac{d}{dt}\sin\theta &= 0 \\
\frac{d}{dt}\cos\theta &= 0
\end{aligned}$$

Now by substituting the above results in the above equations we get the below results:

$$\ddot{x} = \frac{1}{M}\left[F - g\left(m_1\theta_1 + m_2\theta_2\right)\right]$$
$$\ddot{\theta}_1 = \frac{1}{l_1}\left[\frac{F - g\left(m_1\theta_1 + m_2\theta_2\right)}{M} - g\theta_1\right]$$
$$\ddot{\theta}_2 = \frac{1}{l_2}\left[\frac{F - g\left(m_1\theta_1 + m_2\theta_2\right)}{M} - g\theta_2\right]$$

We check the equilibrium condition by considering the given conditions x = 0 , $\theta_1 = 0$ and $\theta_2 =$ 0.In order to represent our linearized system in state space representation we take state variables such that it covers all the dynamics of the system.

The below state variables are being considered to represent our state in state space representation.

$$X(t) = \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}$$

We use the above equations , equilibrium conditions and our state variables to formulate the state space representation which is provided below :

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -g\frac{m_1}{M} & 0 & -g\frac{m_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{g}{l_1}\left(1+\frac{m_1}{M}\right) & 0 & -\frac{g}{l_1}\frac{m_2}{M} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{g}{l_2}\frac{m_1}{M} & 0 & -\frac{g}{l_2}\left(1+\frac{m_2}{M}\right) & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{l_1 M} \\ 0 \\ \frac{1}{l_2 M} \end{bmatrix} u(t)$$

The above system is in the form of

$$\dot{X}(t) = AX(t) + Bu(t)Y(t) = C_X(t) + Du(t)$$

Here $X(t)$ and $u(t)$ are column vectors, A and B are matrix representing the dynamics of the system.

$$\underset{6\times1}{X(\dot{t})} = \underset{6\times1}{AX(t)} + \underset{6\times1}{Bu(t)}$$

A $-_{6\times6}$ Matrix which constiutes all the dynamics of the system.
B $-_{6\times1}$ Matrix which constitutes all the input variables
$X(t)$ $-_{6\times1}$ It consists the state variables
$u(t)$ $-_{1\times1}$ It is the input given to the cart i.e F
C $-_{6\times1}$ Identity Matrix
D - $_{6\times1}$ Zero Matrix

The u(t) is the force **F** exerted on the cart.

1.3 Controllabilty of the linearized state space system

Considering our linearised state system, we now check for the controllabilty of the system.

We have different methods to check the controllabilty of a system, but here we go with the PBH test (Popov Belevitch Hautus) which tells us whether a system is controllable or not.

The PBH test is provided below:

$$[(\lambda I - A) \mid B] \tag{8}$$

We find the rank of the eq:11 and if the rank is a full rank then the system is considered to be controllable else it is not controllable if the rank is less than 'n'.

λ - These are the eigen values of the A

I - Identity matrix

n - The full rank of the system (in our system it is 6 as we have six linearly independent columns)

The system is said to be controllable $\forall \lambda \in C$.Given that all the load mass, lengths of the cables are greater than 0.

$M, m_1, m_2, l_1, l_2 > 0$

$$\lambda I - A = \begin{bmatrix} \lambda & -1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & \frac{g\,m_1}{M} & 0 & \frac{g\,m_2}{M} & 0 \\ 0 & 0 & \lambda & -1 & 0 & 0 \\ 0 & 0 & \frac{g\,(M+m_1)}{M\,l_1} & \lambda & \frac{g\,m_2}{M\,l_1} & 0 \\ 0 & 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & \frac{g\,m_1}{M\,l_2} & 0 & \frac{g\,(M+m_2)}{M\,l_2} & \lambda \end{bmatrix}$$

Now we do the PBH test to find the rank of the below matrix. We found out the rank of this system is 6, which implies that the system is controllable.

$$[(\lambda I - A) \mid B] = \begin{bmatrix} \lambda & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & \frac{g\,m_1}{M} & 0 & \frac{g\,m_2}{M} & 0 & \frac{1}{M} \\ 0 & 0 & \lambda & -1 & 0 & 0 & 0 \\ 0 & 0 & \frac{g\,(M+m_1)}{M\,l_1} & \lambda & \frac{g\,m_2}{M\,l_1} & 0 & \frac{1}{M\,l_1} \\ 0 & 0 & 0 & 0 & \lambda & -1 & 0 \\ 0 & 0 & \frac{g\,m_1}{M\,l_2} & 0 & \frac{g\,(M+m_2)}{M\,l_2} & \lambda & \frac{1}{M\,l_2} \end{bmatrix}$$

We now find the rank of the controllability matrix to check for controllability . The controllablity matrix **C** is given below:

$$\mathbf{C} = \begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B \end{bmatrix}$$

We first check the determinant of this matrix and deduce the conditions for the variables where the system cannot be controllable.

The determinant of the controllabilty matrix is given below.

$$det(C) = -\frac{g^6\,({l_1}^2 - 2\,l_1\,l_2 + {l_2}^2)}{M^6\,{l_1}^6\,{l_2}^6}$$

As we can see that the if the determinant of C is zero , the system is not controllable. Now we equate the determinant of **C** to zero and find the condition where the system is uncontrollable, which can be found out by,

$${l_1}^2 - 2\,l_1\,l_2 + {l_2}^2 = 0 \text{ or } g = 0$$

As g cannot be zero, the above matrix will have a zero determinant only when $l1 = l2$

Hence, the controllability condition is that

$$l1 \neq l2$$

1.4 LQR(Linear Quadratic Regualtor)

Now we design a LQR Controller with the given variables. M = 1000Kg, m1 = m2 = 100Kg, l1 = 20m, l2 = 10m and g = 9.8.We substitute all the variables in our state space system which is given below:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{981}{1000} & 0 & -\frac{981}{1000} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{10791}{20000} & 0 & -\frac{981}{20000} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{981}{10000} & 0 & -\frac{10791}{10000} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{1000} \\ 0 \\ \frac{1}{20000} \\ 0 \\ \frac{1}{10000} \end{bmatrix} u(t)$$

Now we find the eigen values of the the A matrix to check for stability which are given below:

$$\text{eig(A)} = \begin{bmatrix} .0000 + 1.0430i \\ 0.0000 + 1.0430i \\ -0.0000 + 0.7285i \\ -0.0000 - 0.7285i \\ 0.0000 + 0.0000i \\ 0.0000 + 0.0000i \end{bmatrix}$$

As we can see from the above equation , we have zero values for the real part which implies the system is stable.

Now we check the controllabilty of the system by finding the rank of C.

$$\mathbf{C} = \begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B \end{bmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 0 & \frac{1}{1000} & 0 & -\frac{147}{1000000} & 0 & \frac{141659}{1000000000} \\ \frac{1}{1000} & 0 & -\frac{147}{1000000} & 0 & \frac{141659}{1000000000} & 0 \\ 0 & \frac{1}{20000} & 0 & -\frac{637}{20000000} & 0 & \frac{453789}{20000000000} \\ \frac{1}{20000} & 0 & -\frac{637}{20000000} & 0 & \frac{453789}{20000000000} & 0 \\ 0 & \frac{1}{10000} & 0 & -\frac{1127}{10000000} & 0 & \frac{1246119}{10000000000} \\ \frac{1}{10000} & 0 & -\frac{1127}{10000000} & 0 & \frac{1246119}{10000000000} & 0 \end{pmatrix}$$

We find that the rank of the Controllability matrix which we found to be a full rank i.e 6 which implies that the system is controllable.

After verifying that the system is controllable we now implement LQR controller for our linearized system by first finding the optimal cost function using ricartti equation.

The cost function for the LQR controller is given below :

$$J(K, X(0)) = \int_0^\infty \left(X^T Q X + u^T R u\right) dt$$

We optimize our cost function by choosing Q and R matrix which will provides us the optimal solution.

The ricatti equation is provided below where we will use this equation to find the P matrix which is later used to find the gain matrix K.

$$P \cdot A + A^T P - PBR^{-1}B^T P = -Q$$

Now we use our gain matrix and find the state feedback equation for our system as given below:

$$u(t) = -KX(t) = -\left(R^{-1}B^T P\right) X(t)$$

After implementing the LQR controller to our linearlized system with the initial response, we get the the following eigen values by choosing Q matrix as a identity 6 X 1 matrix. First we find out the ouput response of the non-linear system.

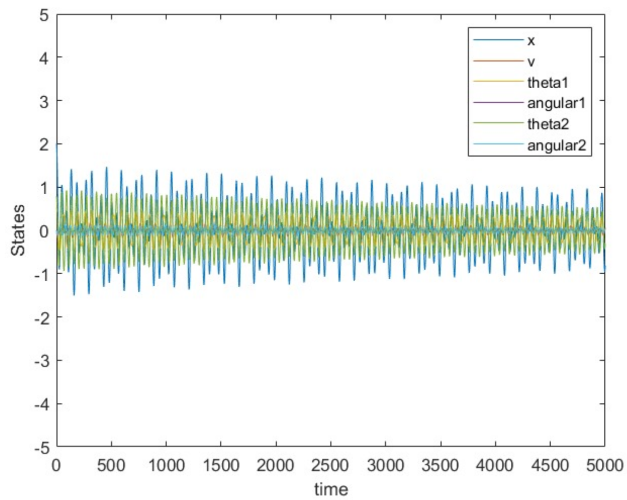

Figure 4: Non-Linear system with initial Q and R

We set the Q and R matrix values and then use the lqr solver to solve for the K gain matrix. We get the following K gain matrix

$$\mathbf{K}=\begin{bmatrix} 100 \\ 497.72 \\ -76.60 \\ -551.75 \\ -34.66 \\ -279.45 \end{bmatrix}$$

$$\mathbf{Q}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{R}$=1

Kindly refer fig.4 for the output of this system with the above Q,R and K matrix. We find the eigen values for the linearized system and check for controllability which is given below:

$$\mathbf{Eig(A\ -}\ B_k K\mathbf{)}=\begin{bmatrix} -0.0090+1.0416i \\ -0.0090-1.0416i \\ -0.0055+0.7273i \\ -0.0055-0.7273i \\ -0.2067+0.2024i \\ -0.2067-0.2024i \end{bmatrix}$$

As we can see all the eigen values of the A matrix lies in the left plane of the real axis which signifies that the system is stable.

Now we again iterate over different values for Q and R matrix in order to decrease the oscillations and the time required to reach the desired state. We have choosen the below values for the Q and R matrix to get the optimal solution.

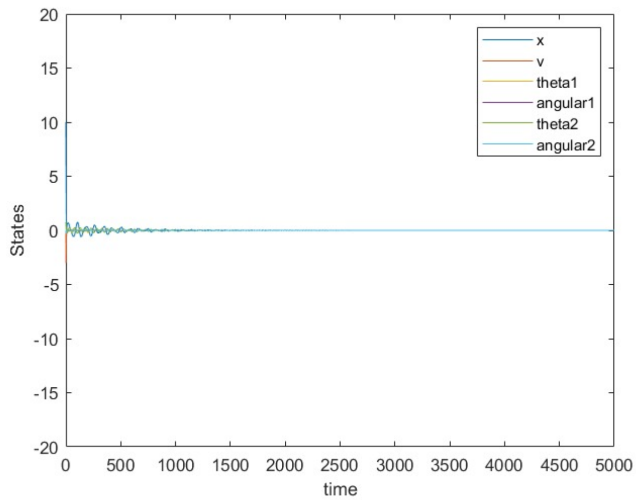

Figure 5: Non-Linear system with optimized Q and R

$$\mathbf{Q}=\begin{bmatrix}5 & 0 & 0 & 0 & 0 & 0\\0 & 1000 & 0 & 0 & 0 & 0\\0 & 0 & 10 & 0 & 0 & 0\\0 & 0 & 0 & 10 & 0 & 0\\0 & 0 & 0 & 0 & 10 & 0\\0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

$\mathbf{R}$ =0.01

The gain matrix we got after calculating it using the ricatti equation (lqr method in MATLAB) is given below :

$$\mathbf{K}= 1.0e{+}03 * \begin{bmatrix}.3162\\1.0151\\2.2875\\0.8244\\0.3704\\-0.3868\end{bmatrix}$$

Kindly refer fig.5 for the output of this system with the above Q,R and K matrix.

$$\mathbf{Eig(A\ -}\ B_kK\mathbf{)}=\begin{bmatrix}-0.0090+1.0416i\\-0.0090-1.0416i\\-0.0055+0.7273i\\-0.0055-0.7273i\\-0.2067+0.2024i\\-0.2067-0.2024i\end{bmatrix}$$

We have reached a optimum state after several trials with the values of Q and R. We found that our values reach the desired state at t = 1000s.Using this lqr controller we have minimized our cost function to get a optimal solution rather than a best solution.

Linearized system output results for LQR controller The initial Q and R matrices for the LQR controller.

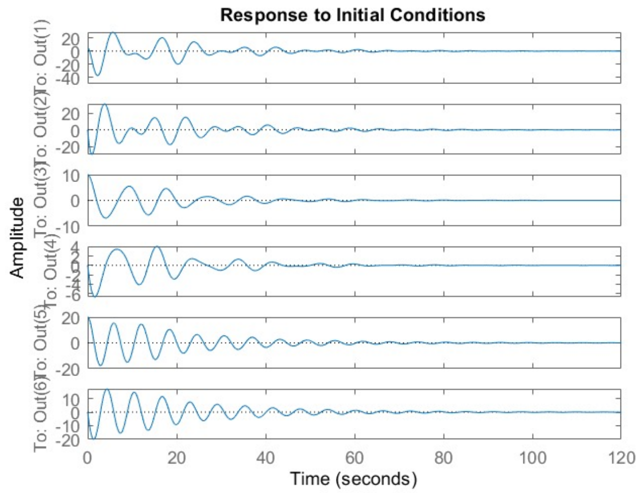

Figure 6: Linear system with initial Q and R

$$\mathbf{Q}=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}$$

$\mathbf{R}$=1

The optimal R and Q matrices have found after tuning these values to get the optimal result.

$$\mathbf{Q}=\begin{bmatrix}10&0&0&0&0&0\\0&5&0&0&0&0\\0&0&1000&0&0&0\\0&0&0&1000&0&0\\0&0&0&0&100&0\\0&0&0&0&0&10\end{bmatrix}$$

$\mathbf{R}$ =0.0001

We have tuned the values and verified its stability using the eigen values of A - $B_k K$ matrix.

We have also simulated our response with linearized system and produced the results above.

By using the lyapunov indirect method stability method , we can check if a system is stable or not based upon the eigen values or poles of the system. If the system which is linearized and is stable at the equillibrium condition then the non-linear state will also be stable locally about the equillibrium point.

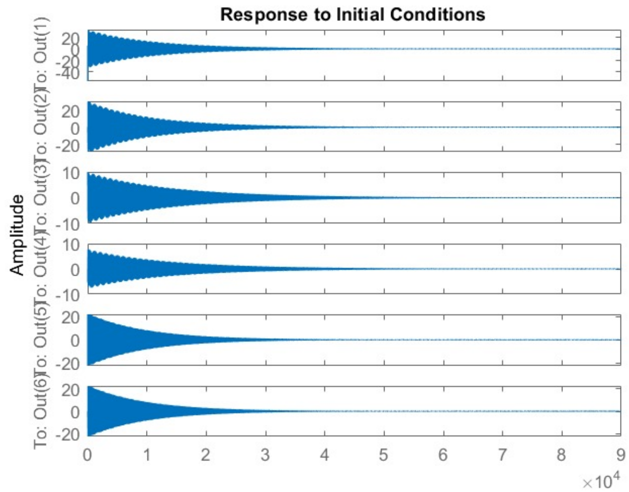

Figure 7: Linear system with optimal initial Q and R

2 Second Component

Until now we have formulated our system's controller to optimize the cost function and to find a desired K gain matrix which will take my state to the desired position. Now we are intended to find a LQG controller where we estimate our own states and feed it to the gain controller to output a optimal solution.

First we find the observability of the below output vectors x(t), (1(t),2(t)), (x(t),2(t)) and (x(t),1(t),2(t)).

To find the observability of the output vector we use the below matrix:

$$\mathbf{O}=\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \\ CA^5 \end{bmatrix}$$

We have checked the observability for the above output vectors and the results are provided below:

$$\mathbf{C1}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{C2}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{C3}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{C4}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The Matlab code for finding the observability can be seen below:

```
clc
clear
```

```
disp("Question E")

syms M m1 m2 l1 l2 g;

A=[0 1 0 0 0 0;
    0 0 -(m1*g)/M 0 -(m2*g)/M 0;
    0 0 0 1 0 0;
    0 0 -((M+m1)*g)/(M*l1) 0 -(m2*g)/(M*l1) 0;
    0 0 0 0 0 1;
    0 0 -(m1*g)/(M*l2) 0 -(g*(M+m2))/(M*l2) 0];

B=[0; 1/M; 0; 1/(M*l1); 0; 1/(M*l2)];


C1 = [1 0 0 0 0 0];
C2 = [0 0 1 0 0 0; 0 0 0 0 1 0];
C3 = [1 0 0 0 0 0; 0 0 0 0 1 0];
C4 = [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];

O1 = [C1; C1*A; C1*A^2; C1*A^3; C1*A^4; C1*A^5];
rank(O1)

O2 = [C2; C2*A; C2*A^2; C2*A^3; C2*A^4; C2*A^5];
rank(O2)

O3 = [C3; C3*A; C3*A^2; C3*A^3; C3*A^4; C3*A^5];
rank(O3)

O4 = [C4; C4*A; C4*A^2; C4*A^3; C4*A^4; C4*A^5];
rank(O4)
```

```
Question E

ans =

     6


ans =

     4


ans =

     6


ans =

     6
```

2.1 Luenberger Observer

We are now going to use the above observer in order to estimate our states.

The state space representation of the Luenberger Observer is given below:

$$\dot{\hat{X}} = A\hat{X} + Bu + L(Y - \hat{Y})$$

The estimation error for the luenberger observer is $\varepsilon = X - \hat{X}$.

$$\dot{X}(t) = AX + BU$$

Now we find the state space representation for the estimation error.

$$\dot{\varepsilon} = \dot{X} - \dot{\hat{X}} = (AX + BU) - (A\hat{X} + BU + L(Y - \hat{Y})) \quad = AX - A\hat{X} - L(CX - C\hat{X}) = (A - LC)\varepsilon$$

After simulating the Leunberger observer for the given system, using the provided output vector, we were able to optimize the observer for the three observable systems for corresponding output vectors. Following initial condition and unit step responses of the system were simulated.

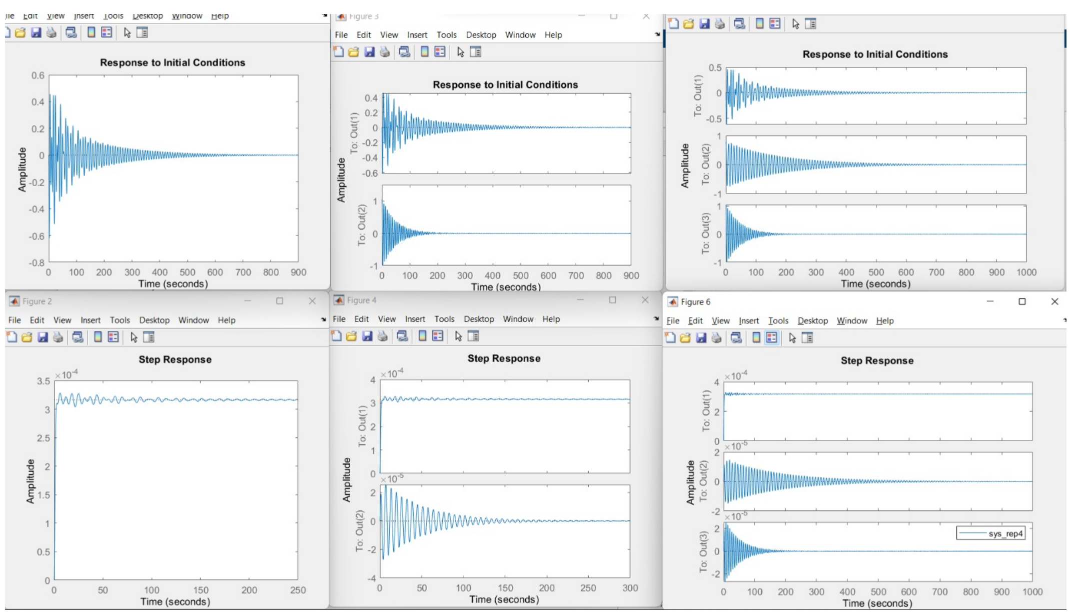

Figure 8: Linear System for Question F

2.2 Appendix

Please refer to the Github Repository:

2.3 Question C

```
clc
clear

disp("Question C")

syms M m1 m2 l1 l2 g;

A=[0 1 0 0 0 0;
    0 0 -(m1*g)/M 0 -(m2*g)/M 0;
    0 0 0 1 0 0;
    0 0 -((M+m1)*g)/(M*l1) 0 -(m2*g)/(M*l1) 0;
    0 0 0 0 0 1;
    0 0 -(m1*g)/(M*l2) 0 -(g*(M+m2))/(M*l2) 0];

B=[0; 1/M; 0; 1/(M*l1); 0; 1/(M*l2)];

% Ctrl = ctrb(A, B)
Ctrl = [B A*B (A^2)*B (A^3)*B (A^4)*B (A^5)*B];


disp("The rank of the controllability matrix is:")
rank(Ctrl)

disp("The determinant of the controllability matrix is:")
disp(det(Ctrl));


lamda = eye(6,6) * sym('lambda');


A_lbd = [lamda - A B];

rank(A_lbd);
```

```
Question C
The rank of the controllability matrix is:

ans =

     6

The determinant of the controllability matrix is:
-(g^6*l1^2 - 2*g^6*l1*l2 + g^6*l2^2)/(M^6*l1^6*l2^6)
```

2.4 Question D

```
clc
clear
```

****

```
disp("Question D")

syms M m1 m2 l1 l2 g;

A=[0 1 0 0 0 0;
    0 0 -(m1*g)/M 0 -(m2*g)/M 0;
    0 0 0 1 0 0;
    0 0 -((M+m1)*g)/(M*l1) 0 -(m2*g)/(M*l1) 0;
    0 0 0 0 0 1;
    0 0 -(m1*g)/(M*l2) 0 -(g*(M+m2))/(M*l2) 0];

A = double(subs(A, {M, m1, m2, l1, l2, g}, {1000, 100, 100, 20, 10, 9.8}));

B=[0; 1/M; 0; 1/(M*l1); 0; 1/(M*l2)];

B = double(subs(B, {M, m1, m2, l1, l2, g}, {1000, 100, 100, 20, 10, 9.8}));

% Ctrl = ctrb(A, B)
Ctrl = ([B A*B (A^2)*B (A^3)*B (A^4)*B (A^5)*B]);

disp("The eigen value of the controllability matrix is:")

eigs(double(A))

disp("The rank of the controllability matrix is:")
rank(Ctrl)

lamda = eye(6,6) * sym('lambda');


A_lbd = [lamda - A B];

rank(A_lbd);

intial_state = [5;0;10;0;20;0];

Q=[10 0 0 0 0 0;
   0 5 0 0 0 0;
   0 0 1000 0 0 0;
   0 0 0 1000 0 0;
   0 0 0 0 100 0;
   0 0 0 0 0 10];
R = 0.0001;

C = eye(6);
D = 0;

sys_rep = ss(A,B,C,D);

K_val = lqr(A,B,Q,R);
```

****

```
sys_rep_k = ss(A-B*K_val,B,C,D);

eigs(A-B*K_val)

figure
initial(sys_rep_k,intial_state)
```

```
Question D
The eigen value of the controllability matrix is:

ans =

   0.0000 + 1.0425i
   0.0000 - 1.0425i
  -0.0000 + 0.7282i
  -0.0000 - 0.7282i
   0.0000 + 0.0000i
   0.0000 + 0.0000i

The rank of the controllability matrix is:

ans =

     6


ans =

  -0.0555 + 1.0310i
  -0.0555 - 1.0310i
  -0.1015 + 0.7238i
  -0.1015 - 0.7238i
  -0.3613 + 0.3690i
  -0.3613 - 0.3690i
```

2.5 Question D - non linear

```
clear
clc

M=1000;
m1=100;
m2=100;
l1=20;
l2=10;
g=9.8;

A=[0 1 0 0 0 0;
    0 0 -(m1*g)/M 0 -(m2*g)/M 0;
    0 0 0 1 0 0;
```

```
    0 0 -((M+m1)*g)/(M*l1) 0 -(m2*g)/(M*l1) 0;
    0 0 0 0 0 1;
    0 0 -(m1*g)/(M*l2) 0 -(g*(M+m2))/(M*l2) 0];

B=[0; 1/M; 0; 1/(M*l1); 0; 1/(M*l2)];

% Q=[10 0 0 0 0 0;
%    0 100 0 0 0 0;
%    0 0 0 10 0 0;
%    0 0 0 100 0 0;
%    0 0 0 0 10 0;
%    0 0 0 0 0 10000];
% R = 0.01;
Q=[1 0 0 0 0 0;
   0 1 0 0 0 0;
   0 0 1 0 0 0;
   0 0 0 1 0 0;
   0 0 0 0 1 0;
   0 0 0 0 0 1];
R = 1;

initial_x = [2; 0; pi/6; 0; pi/4; 0];
tspan = 0:0.1:5000;
rank(ctrb(A,B))
[K_val, ~, ~] = lqr(A,B,Q,R);

F=@(x)-K_val*x;
eigs(A-B*K_val)

[final_t, final_x] = ode45(@(t, x)cart_system(x, M, m1, m2, l1, l2, g, F(x)), tspan, initial_x);
plot(final_t, final_x)
ylim([-5, 5])
xlabel("time");
ylabel("States")
legend('x', 'v', 'theta1', 'angular1', 'theta2', 'angular2')

function dx = cart_system(x, M, m1, m2, l1, l2, g, F)



dx=zeros(6,1);
dx(1) = x(2);
dx(2)=(F-(g/2)*(m1*sind(2*x(3))+m2*sind(2*x(5)))-(m1*l1*(x(4)^2)*sind(x(3)))-(m2*l2*(x(6)^2)*sind(x(5
dx(3)= x(4);
dx(4)= (dx(2)*cosd(x(3))-g*(sind(x(3))))/l1';
dx(5)= x(6);
dx(6)= (dx(2)*cosd(x(5))-g*(sind(x(5))))/l2;
end

ans =

     6
```

```
ans =

  -0.0001 + 1.0425i
  -0.0001 - 1.0425i
  -0.0001 + 0.7282i
  -0.0001 - 0.7282i
  -0.0204 + 0.0204i
  -0.0204 - 0.0204i
```

2.6 Question F - linear

```
clc
clear

disp("Question F")

syms M m1 m2 l1 l2 g;

A=[0 1 0 0 0 0;
   0 0 -(m1*g)/M 0 -(m2*g)/M 0;
   0 0 0 1 0 0;
   0 0 -((M+m1)*g)/(M*l1) 0 -(m2*g)/(M*l1) 0;
   0 0 0 0 0 1;
   0 0 -(m1*g)/(M*l2) 0 -(g*(M+m2))/(M*l2) 0];

A = double(subs(A, {M, m1, m2, l1, l2, g}, {1000, 100, 100, 20, 10, 9.8}));

B=[0; 1/M; 0; 1/(M*l1); 0; 1/(M*l2)];

B = double(subs(B, {M, m1, m2, l1, l2, g}, {1000, 100, 100, 20, 10, 9.8}));

% Ctrl = ctrb(A, B)
Ctrl = ([B A*B (A^2)*B (A^3)*B (A^4)*B (A^5)*B]);

C1 = [1 0 0 0 0 0; 0 0 0 0 0 0;0 0 0 0 0 0];
C2 = [0 0 1 0 0 0; 0 0 0 0 1 0;0 0 0 0 0 0];
C3 = [1 0 0 0 0 0; 0 0 0 0 1 0;0 0 0 0 0 0];
C4 = [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];

D = 0;

Q=[10000 0 0 0 0 0;
   0 1000 0 0 0 0;
   0 0 100 0 0 0;
   0 0 0 10 0 0;
   0 0 0 0 1000 0;
   0 0 0 0 0 100];
R = 0.001;
```

```
new_poles=[-0.1;-0.3;-0.5;-0.7;-0.9;-1.1];

K=lqr(A,B,Q,R);

x_initial = [0, 0, pi/4, 0, pi/3, 0, 0, 0, 0, 0, 0, 0];

Luenberger_B = [B;zeros(size(B))];

L1 = place(A',C1',new_poles)';
Luenberger_A1 = [(A-B*K) B*K;
        zeros(size(A)) (A-L1*C1)];
Luenberger_C1 = [C1 zeros(size(C1))];
sys_rep1 = ss(Luenberger_A1,Luenberger_B,Luenberger_C1,D);
figure
initial(sys_rep1,x_initial)
figure
step(sys_rep1)

L3 = place(A',C3',new_poles)';
Luenberger_A3 = [(A-B*K) B*K;
        zeros(size(A)) (A-L3*C3)];
Luenberger_C3 = [C3 zeros(size(C3))];
sys_rep3 = ss(Luenberger_A3,Luenberger_B,Luenberger_C3,D);
figure
initial(sys_rep3,x_initial)
figure
step(sys_rep3)

L4 = place(A',C4',new_poles)';
Luenberger_A4 = [(A-B*K) B*K;
        zeros(size(A)) (A-L4*C4)];
Luenberger_C4 = [C4 zeros(size(C4))];
sys_rep4 = ss(Luenberger_A4,Luenberger_B,Luenberger_C4,D);
figure
initial(sys_rep4,x_initial)
figure
step(sys_rep4)
```

2.7 Question G - Linear

```
clc
clear

disp("Question G")

syms M m1 m2 l1 l2 g;

A=[0 1 0 0 0 0;
   0 0 -(m1*g)/M 0 -(m2*g)/M 0;
   0 0 0 1 0 0;
```

```
   0 0 -((M+m1)*g)/(M*l1) 0 -(m2*g)/(M*l1) 0;
   0 0 0 0 0 1;
   0 0 -(m1*g)/(M*l2) 0 -(g*(M+m2))/(M*l2) 0];

A = double(subs(A, {M, m1, m2, l1, l2, g}, {1000, 100, 100, 20, 10, 9.8}));

B=[0; 1/M; 0; 1/(M*l1); 0; 1/(M*l2)];

B = double(subs(B, {M, m1, m2, l1, l2, g}, {1000, 100, 100, 20, 10, 9.8}));

C1 = [1 0 0 0 0 0];
C2 = [0 0 1 0 0 0; 0 0 0 0 1 0];
C3 = [1 0 0 0 0 0; 0 0 0 0 1 0];
C4 = [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];

D = 0;

Q=[10000 0 0 0 0 0;
   0 1000 0 0 0 0;
   0 0 100 0 0 0;
   0 0 0 10 0 0;
   0 0 0 0 1000 0;
   0 0 0 0 0 100];
R = 0.05;

new_poles=[-0.1;-0.3;-0.5;-0.7;-0.9;-1.1];

K=lqr(A,B,Q,R);

x_initial = [0, 0, pi/4, 0, pi/3, 0, 0, 0, 0, 0, 0, 0];

Vd = 0.1*eye*(6);
Vn = 0.5;

K_C1 = lqr(A', C1', Vd, Vn)';
sys_rep1 = ss([(A-B*K) B*K; zeros(size(A)) (A-K_C1*C1)], [B;zeros(size(B))],[C1 zeros(size(C1))], D);
figure
initial(sys_rep1,x_initial)
figure
step(sys_rep1)

K_C3 = lqr(A', C3', Vd, Vn)';
sys_rep3 = ss([(A-B*K) B*K; zeros(size(A)) (A-K_C3*C3)], [B;zeros(size(B))],[C3 zeros(size(C3))], D);
figure
initial(sys_rep3,x_initial)
figure
step(sys_rep3)

K_C4 = lqr(A', C4', Vd, Vn)';
sys_rep4 = ss([(A-B*K) B*K; zeros(size(A)) (A-K_C4*C4)], [B;zeros(size(B))],[C4 zeros(size(C4))], D);
figure
initial(sys_rep4,x_initial)
```

****

```
figure
step(sys_rep4)
```

Question G

```
clear
clc

syms M m1 m2 l1 l2 g;

A=[0 1 0 0 0 0;
    0 0 -(m1*g)/M 0 -(m2*g)/M 0;
    0 0 0 1 0 0;
    0 0 -((M+m1)*g)/(M*l1) 0 -(m2*g)/(M*l1) 0;
    0 0 0 0 0 1;
    0 0 -(m1*g)/(M*l2) 0 -(g*(M+m2))/(M*l2) 0];

B=[0; 1/M; 0; 1/(M*l1); 0; 1/(M*l2)];

A = double(subs(A, {M, m1, m2, l1, l2, g}, {1000, 100, 100, 20, 10, 9.8}));

B = double(subs(B, {M, m1, m2, l1, l2, g}, {1000, 100, 100, 20, 10, 9.8}));

M=1000;
m1=100;
m2=100;
l1=20;
l2=10;
g=9.81;

Q=[10 0 0 0 0 0;
   0 100 0 0 0 0;
   0 0 10 0 0 0;
   0 0 0 100 0 0;
   0 0 0 0 10 0;
   0 0 0 0 0 10000];
R = 0.01;

initial_x = [0; 0; pi/6; 0; pi/4; 0; 0; 0; 0; 0; 0; 0];
tspan = 0:0.1:5000;

[final_t, final_x] = ode45(@(t, x)cart_system(x, M, m1, m2, l1, l2, g, A, B, Q, R), tspan, initial_x)
hold on
plot(final_t, final_x)
% ylim([-5, 5])
xlabel("time");
ylabel("States")
legend('x', 'v', 'theta1', 'angular1', 'theta2', 'angular2')
hold off

function dx = cart_system(x, M, m1, m2, l1, l2, g, A, B, Q, R)
```

****

```
C1 = [1 0 0 0 0 0;0 0 0 0 0 0;0 0 0 0 0 0];

[K, ~, ~] = lqr(A,B,Q,R);
F=-K*x(1:6);

Vd=0.1*eye(6);
Vn=5;
Kp = lqr(A',C1',Vd,Vn)';

S =(A-Kp*C1)*x(7:12);

dx=zeros(12,1);
dx(1) = x(2);
dx(2)=(F-(g/2)*(m1*sind(2*x(3))+m2*sind(2*x(5)))-(m1*l1*(x(4)^2)*sind(x(3)))-(m2*l2*(x(6)^2)*sind(x(5
dx(3)= x(4);
dx(4)= (dx(2)*cosd(x(3))-g*(sind(x(3))))/l1';
dx(5)= x(6);
dx(6)= (dx(2)*cosd(x(5))-g*(sind(x(5))))/l2;
dx(7)= x(2)-x(10);
dx(8)= dx(2)-S(2);
dx(9)= x(4)-x(11);
dx(10)= dx(4)-S(4);
dx(11)= x(6)-x(12);
dx(12)= dx(6)-S(6);
end
```